Computation for Jacobi-Gauss Lobatto Quadrature Based on Derivative Relation

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Abstract. The three-term recurrence relation for derivatives of Jacobi-type polynomial is derived and the Gauss-Lobatto points are also the eigenvalues of some special Jacobi matrix instead of G.H. Golub's modified Jacobi matrix. In addition, explicit weights are also presented much more simply. Numerical experiments are carried out to illustrate the effectiveness of the method, and the comparison is also given to demonstrate that it's more robust and accuracy based on three-term recurrence relation for derivatives.

Key words. Three-term recurrence relation for derivatives; Jacobi-Gauss Lobatto quadrature; Modified Jacobi matrix

AMS subject classifications. 41A55, 65D32

1. Introduction. In many science and engineering applications, PDEs will be involved with high precision quadrature on the reference element after some numerical discretization like FEM. Gauss-type quadrature is appreciated by practical engineers with its high accuracy. Many years ago, it had been proved that the Gauss points were exactly the zeros of some orthogonal polynomials with respect to its nonnegative weight function (cf.[15]). But until 1969, G.H Golub[2] successfully changed the problem of locating zeros of orthogonal polynomials into computing the eigenvalues of a tridiagonal Jacobi matrix. The method for computing (n+2)-points Gauss-Lobatto quadrature rule for any measure of integration is to generate the Jacobi matrix of order n+2 for the measure firstly, then modify the three elements at the right lower corner of the matrix in a manner proposed by Golub, and finally compute the eigenvalues and the first component of the respective eigenvectors to produce the nodes and weights of the quadrature rule. This work has been done by G.H. Golub[1] in 1973. Generally, it works quite well, but when n becomes large, underflow problem causes the method to fail. This crucial phenomenon was first reported by Walter Cautschi[4]. To avoid the underflow problem, Cautschi computed the modified elements directly for arbitrary Jacobi measures instead of solving the 2×2 system of linear equations that was used to compute the modified Jacobi matrix elements and gave explicit expression of all the weights of the quadrature rules. The work of Gauss-Lobatto formulae seems to be an end, however, It is of course a good idea mathematically to compute the modified Jacobi matrix to get the Gauss-Lobatto nodes, but not directly, and the explicit formulae of weights is not so elegantly but somewhat complicated.

The aim of the paper is to derive the three-term recurrence relation of the derivatives of Jacobi polynomials which is new to our best knowledge, and compute eigenvalues of the special Jacobi matrix of order n which is different from the modified Jacobi matrix proposed by Golub. Then we develop another more simple explicit expression of weights of the quadrature rule which is quite practical in the actual computation.

The rest of the paper is organized as follows. In Sect.2, we simply recall some crucial results proposed by G.H. Golub and improved later by Walter Cautschi. In Sect3, the three-term recurrence relation for the derivatives of Jacobi polynomial is derived. In the following, explicit formula for weights of the quadrature rule is presented for arbitrary Jacobi measures. Numerical comparisons and concluding remarks are given in the final section.

2. Some basis results for Gauss-Lobatto quadrature formulae. Given a positive measure $d\lambda$ supported on the interval [-1,1], and assume that all its moments exist, We restrict our attention to Gauss-Lobatto quadrature of the form

$$\int_{-1}^{1} f(x)d\lambda(x) = \lambda_0 f(-1) + \sum_{k=1}^{n} \lambda_k f(x_k) + \lambda_{n+1} f(1) + R_n(f), \tag{2.1}$$

which is exact whenever f is a polynomial of degree $\le 2n+1$,

$$R_n(f) = 0 \qquad \forall\, f \in \mathrm{P}_{2n+1}$$

It is called Gauss-Lobatto rule with ($n+2$) points relative to the measure $d\lambda$. As is well known, the interior nodes x_k are the zeros of $\pi_n(\cdot\,;d\lambda_{\pm 1})$ or the polynomials of degree n orthogonal with respect to the modified measure $d\lambda_{\pm 1}(x) = (1-x^2)d\lambda(x)$, where $\pi_n(\cdot\,;d\lambda)$ are (monic) orthogonal polynomial of degree n relative to measure $d\lambda$ and satisfy the following three-term recurrence relation

$$\pi_{k+1}(x) = (x-\alpha_k)\pi_k(x) - \beta_k \pi_{k-1}(x), \quad k = 0,1,2,\cdots,n-1 \tag{2.2}$$

with $\pi_{-1}(x) = 0, \pi_0(x) = 1$, $\alpha_k = \alpha_k(d\lambda) \in \mathbb{R}, \beta_k = \beta_k(d\lambda) > 0$ and $\beta_0 = \int_{-1}^{1} d\lambda(x)$.

The Jacobi matrix of order n is defined by

$$J_n^G = J_n^G(d\lambda) = \begin{pmatrix} \alpha_0 & \sqrt{\beta_1} & & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \ddots & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ 0 & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{pmatrix} \tag{2.3}$$

Lemma 2.1 (Golub, Welsch[2]**)** The Gauss nodes x_k^G are the eigenvalues of J_n^G and the Gauss weights λ_k^G are given by

$$\lambda_k^G = \beta_0 (u_{k,1}^G)^2, \qquad k = 1,2,3,\cdots,n \tag{2.4}$$

where u_k^G is the normalized eigenvector of J_n^G corresponding to the eigenvalues x_k^G (i.e. $(u_k^G)^T u_k^G = 1$) and $u_{k,1}^G$ is its first component.

For Gauss-Lobatto formula (2.1) the Jacobi-Lobatto matrix of order n+2 is modified as

$$J_{n+2}^L = J_{n+2}^L(d\lambda) = \left[\begin{array}{c|cc} J_n^G(d\lambda) & \sqrt{\beta_n} e_n & 0 \\ \hline \sqrt{\beta_n} e_n^T & \alpha_n & \sqrt{\beta_{n+1}^L} e_{n+1} \\ 0 & \sqrt{\beta_{n+1}^L} e_{n+1}^T & \alpha_{n+1}^L \end{array}\right] \tag{2.5}$$

where $J_n^G(d\lambda)$ is as before, and $\alpha_{n+1}^L, \beta_{n+1}^L$ is the solution of the 2×2 linear system

$$\begin{bmatrix} \pi_{n+1}(-1) & \pi_n(-1) \\ \pi_{n+1}(1) & \pi_n(1) \end{bmatrix} \begin{bmatrix} \alpha_{n+1}^L \\ \beta_{n+1}^L \end{bmatrix} = \begin{bmatrix} -\pi_{n+1}(-1) \\ \pi_{n+1}(1) \end{bmatrix} \tag{2.6}$$

so we have

Lemma 2.2 (Golub[1]**)** The Gauss-Lobatto nodes $x_0^L = -1, x_1^L, \cdots, x_n^L, x_{n+1}^L = 1$ are the eigenvalues of J_{n+2}^L and the Gauss-Lobatto weights λ_k^L are given by

$$\lambda_k^L = \beta_0 (u_{k,1}^L)^2, \qquad k = 0,1,2,3,\cdots,n,n+1 \tag{2.7}$$

where u_k^L is the normalized eigenvector of J_{n+2}^L corresponding to the eigenvalues x_k^L (i.e. $(u_k^L)^T u_k^L = 1$) and $u_{k,1}^L$ is its first component.

Gauss-Lobatto formulae are therefore computable by the QR algorithm applied to J_{n+2}^L. But when *n* becomes large, the elements of the matrix (2.6) become very small and the products is even smaller, this will create a singular system. For the Legendre measure $d\lambda(x) = dx$ and single-precision IEEE arithmetic, this happens beginning with *n*=79, and in double precision, beginning with *n*=543. This crucial phenomenon was first reported by Walter Cautschi[4]. In order to avoid underflow problem, Cautschi directly computed the modified elements for arbitrary Jacobi measures instead of solving the system (2.6), and meanwhile he also derived the explicit expression of associated weights. Numerical results showed that his method is more superior to Golub's.

3. The relation for derivative of Jacobi polynomials. We consider Jacobi polynomials denoted by $J_n^{\alpha,\beta}(x)$ with respect to weight function

$$\omega(x) = (1-x)^\alpha (1+x)^\beta \qquad \alpha, \beta > -1 . \tag{3.1}$$

As is well known, $J_n^{\alpha,\beta}(x)$ are the solutions of singular Sturm-Liouville problem:

$$(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d}{dx}\left\{(1-x)^{1+\alpha}(1+x)^{1+\beta}\frac{d}{dx}J_n^{\alpha,\beta}(x)\right\}+n(n+1+\alpha+\beta)J_n^{\alpha,\beta}(x)=0, \tag{3.2}$$

which can be rewritten as

$$(1-x^2)\frac{d^2}{dx^2}J_n^{\alpha,\beta}(x)+\left(\beta-\alpha-(\beta+\alpha+2)x\right)\frac{d}{dx}J_n^{\alpha,\beta}(x)+n(n+1+\alpha+\beta)J_n^{\alpha,\beta}(x)=0. \tag{3.3}$$

It is easy to have from (3.2)

$$\int_{-1}^{1}(1-x)^{1+\alpha}(1+x)^{1+\beta}\frac{d}{dx}J_n^{\alpha,\beta}(x)\frac{d}{dx}J_m^{\alpha,\beta}(x)=0, \quad m \neq n ,$$

which implies that $\frac{d}{dx}J_n^{\alpha,\beta}(x)$ are also mutually orthogonal with respect to the weight function $\omega(x) = (1-x)^{1+\alpha}(1+x)^{1+\beta}$, so $\frac{d}{dx}J_n^{\alpha,\beta}(x)$ must take the form (2.2).

This motivates us to seek for the corresponding α_k, β_k .

Recall that Jacobi polynomials satisfy the following three-term recurrence relation (cf. Eq.(4.5.1) in Szego's[13])

$$J_{n+1}^{\alpha,\beta}(x) = (D_n x + E_n) J_n^{\alpha,\beta}(x) + F_n J_{n-1}^{\alpha,\beta}(x), n = 1, 2, \ldots \tag{3.4}$$

With $J_0^{\alpha,\beta}(x) = 1$, $J_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta)$,

$$\text{where} \begin{cases} D_n = \dfrac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)} \\ E_n = \dfrac{(\alpha^2-\beta^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)} \\ F_n = -\dfrac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)} \qquad n = 1, 2\ldots \end{cases}. \tag{3.5}$$

Another important formula used frequently[13] is

$$\begin{aligned} (1-x^2)\frac{d}{dx}J_n^{\alpha,\beta}(x) &= \frac{1}{2}(n+\alpha+\beta+1)(1-x^2)J_{n-1}^{\alpha+1,\beta+1}(x) \\ &= A_n J_{n-1}^{\alpha,\beta} + B_n J_n^{\alpha,\beta} + C_n J_{n+1}^{\alpha,\beta} \end{aligned} \tag{3.6}$$

$$\text{where} \begin{cases} A_n = \dfrac{2(n+\alpha)(n+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} \\ B_n = \dfrac{2(\alpha-\beta)n(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} \\ C_n = -\dfrac{2n(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \qquad n = 1, 2\ldots \end{cases}. \tag{3.7}$$

Theorem 3.1 The derivatives of Jacobi polynomials satisfy the following three-term recursive relation

$$\frac{d}{dx}J_{n+1}^{\alpha,\beta}(x) = (a_n x + b_n)\frac{d}{dx}J_n^{\alpha,\beta}(x) - c_n \frac{d}{dx}J_{n-1}^{\alpha,\beta}(x), \quad n \ge 1 \tag{3.8}$$

with $\frac{d}{dx}J_0^{\alpha,\beta}(x) = 0, \frac{d}{dx}J_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+2)$,

$$\text{where} \quad \begin{aligned} a_n &= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2n(n+\alpha+\beta+1)} \\ b_n &= \frac{\left(\alpha^2-\beta^2+2(\alpha-\beta)\right)(2n+\alpha+\beta+1)}{2n(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \\ c_n &= \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{n(n+\alpha+\beta)(2n+\alpha+\beta)} \end{aligned} \tag{3.9}$$

Proof: Differentiate both side of the Eq. (3.4) to obtain

$$\frac{d}{dx}J_{n+1}^{\alpha,\beta}(x) = D_n J_n^{\alpha,\beta}(x) + (D_n x + E_n)\frac{d}{dx}J_n^{\alpha,\beta}(x) + F_n \frac{d}{dx}J_{n-1}^{\alpha,\beta}(x). \tag{3.10}$$

Multiply both side of Eq.(3.6) by $(1-x)^{\alpha}(1+x)^{\beta}$, and differentiate it to get

$$\frac{d}{dx}\left\{(1-x)^{1+\alpha}(1+x)^{1+\beta}\frac{d}{dx}J_{n}^{\alpha,\beta}(x)\right\}=A_{n}\frac{d}{dx}\left(J_{n-1}^{\alpha,\beta}(x)\omega(x)\right)+B_{n}\frac{d}{dx}\left(J_{n}^{\alpha,\beta}(x)\omega(x)\right)+C_{n}\frac{d}{dx}\left(J_{n+1}^{\alpha,\beta}(x)\omega(x)\right). \quad (3.11)$$

It's trivial to show that

$$\frac{\omega'(x)}{\omega(x)}=\left(\ln\omega(x)\right)'=\left(\alpha\ln(1-x)+\beta\ln(1+x)\right)'=-\frac{\alpha}{1-x}+\frac{\beta}{1+x} \quad (3.12)$$

For the left part of Eq. (3.11), it yields using the relation of formula (3.2)

$$\frac{d}{dx}\left\{(1-x)^{1+\alpha}(1+x)^{1+\beta}\frac{d}{dx}J_{n}^{\alpha,\beta}(x)\right\}=-n(n+1+\alpha+\beta)J_{n}^{\alpha,\beta}(x)\omega(x), \quad (3.13)$$

and For the right part of Eq. (3.11), we get by computing directly

$$\begin{aligned}&A_{n}\frac{d}{dx}\left(J_{n-1}^{\alpha,\beta}(x)\omega(x)\right)+B_{n}\frac{d}{dx}\left(J_{n}^{\alpha,\beta}\omega(x)\right)+C_{n}\frac{d}{dx}\left(J_{n+1}^{\alpha,\beta}\omega(x)\right)\\&=\omega(x)\left(A_{n}\frac{d}{dx}J_{n-1}^{\alpha,\beta}(x)+B_{n}\frac{d}{dx}J_{n}^{\alpha,\beta}(x)+C_{n}\frac{d}{dx}J_{n+1}^{\alpha,\beta}(x)+\frac{\omega'(x)}{\omega(x)}\left(A_{n}J_{n-1}^{\alpha,\beta}(x)+B_{n}J_{n}^{\alpha,\beta}(x)+C_{n}J_{n+1}^{\alpha,\beta}(x)\right)\right).\end{aligned}$$

Using the above formulae in combination with (3.6),(3.11), (3.12) and (3.13), the following formula can be derived

$$-n(n+1+\alpha+\beta)J_{n}^{\alpha,\beta}(x)=A_{n}\frac{d}{dx}J_{n-1}^{\alpha,\beta}(x)+B_{n}\frac{d}{dx}J_{n}^{\alpha,\beta}(x)+C_{n}\frac{d}{dx}J_{n+1}^{\alpha,\beta}(x)+\left(-\frac{\alpha}{1-x}+\frac{\beta}{1+x}\right)(1-x^{2})\frac{d}{dx}J_{n}^{\alpha,\beta}(x)$$

which can be simplified into

$$-n(n+1+\alpha+\beta)J_{n}^{\alpha,\beta}(x)=A_{n}\frac{d}{dx}J_{n-1}^{\alpha,\beta}(x)+\left(\left(-(\alpha+\beta)x+B_{n}-\alpha+\beta\right)\frac{d}{dx}J_{n}^{\alpha,\beta}(x)\right)+C_{n}\frac{d}{dx}J_{n+1}^{\alpha,\beta}(x). \quad (3.14)$$

Thanks to (3.10), we can have

$$J_{n}^{\alpha,\beta}(x)=-\frac{1}{D_{n}}\left(-\frac{d}{dx}J_{n+1}^{\alpha,\beta}(x)+(D_{n}x+E_{n})\frac{d}{dx}J_{n}^{\alpha,\beta}(x)+F_{n}\frac{d}{dx}J_{n-1}^{\alpha,\beta}(x)\right),$$

and substitute it into formula (3.14), the final three-term recursive relation for the derivative of Jacobi polynomials can be gotten after complicated computation, which completes the proof.

To make $\frac{d}{dx}J_{n}^{\alpha,\beta}(x)$ be monic, we set $q_{n}=\lambda_{n}\frac{d}{dx}J_{n}^{\alpha,\beta}(x)$ which is of degree n-1. It is can gotten from (3.8)

$$xq_{n}=\frac{1}{a_{n}}\left(\frac{\lambda_{n}}{\lambda_{n+1}}q_{n+1}-b_{n}q_{n}+c_{n}\frac{\lambda_{n}}{\lambda_{n-1}}q_{n-1}\right). \quad (3.15)$$

Therefore, the following should be satisfied (for consistency with (2.2))

$$1=\frac{1}{a_{n}}\frac{\lambda_{n}}{\lambda_{n+1}}, \qquad \alpha_{n}=-\frac{b_{n}}{a_{n}}, \qquad \beta_{n}=\frac{c_{n}}{a_{n}}\frac{\lambda_{n}}{\lambda_{n-1}}.$$

Inferring from the above equation, we have

$$\alpha_n = -\frac{\alpha^2-\beta^2+2(\alpha-\beta)}{(2n+a+b)(2n+a+b+2)}, \beta_n = \frac{4(n-1)(n+\alpha)(n+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)} n \geq 1 \quad (3.16)$$

Remark:

If $\alpha+\beta=-1$, the denominator of β_1 is vanished , but the nominator *n-1* is also zeros, this time we can set $\beta_n = \frac{(n+\alpha)(n+\beta)}{(2n-1)^2}$ directly from (3.16). Though it happens, it makes sense to our Jacobi matrix. In fact, our Jacobi matrix needn't β_1, which can be ignored naturally.

From (3.15) and (3.16), we know that (replacing *n* by *k*)

$$xq_k = q_{k+1} + \alpha_k q_k + \beta_k q_{k-1}) \quad k = 1, 2, \cdots, n-1 \quad (3.17)$$

with $q_0 = 0, q_1 = 1$.

So a special Jacobi matrix of only order *n-1* for the zeros of $\frac{d}{dx} J_n^{\alpha,\beta}(x)$ with the newly-established relation (3.17) is listed as following

$$\hat{J}_{n-1} = \begin{pmatrix} \alpha_1 & \sqrt{\beta_2} & & & 0 \\ \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ 0 & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{pmatrix},$$

and the details of constructing general Jacobi matrix relative to the orthogonal polynomials can be found in [15].

Our Jacobi matrix is somewhat analogous to Gulob's Jacobi matrix for Gauss quadrature nodes instead of modified Jacobi matrix for Gauss- Lobatto quadrature nodes which we have mentioned before.

We recall that the interior nodes x_k of Gauss-Lobatto quadrature (2.1) are the zeros of $\pi_n(\cdot; d\lambda_{\pm 1})$ or the polynomials of degree *n* orthogonal with respect to the modified measure $d\lambda_{\pm 1}(x) = (1-x^2)d\lambda(x)$. For Jacobi weight function $\omega(x) = (1-x)^\alpha(1+x)^\beta$, $d\lambda_{\pm}^{\alpha,\beta}(x) = (1-x^2)\omega(x)dx = d\lambda^{\alpha+1,\beta+1}(x)$, the interior nodes x_k are therefore the zeros of $J_n^{\alpha+1,\beta+1}(x)$. Note that the first equality of Eq.(3.6), so we have

$$\frac{d}{dx} J_{n+1}^{\alpha,\beta}(x) = \frac{1}{2}(n+\alpha+\beta+1) J_n^{\alpha+1,\beta+1}(x)$$

That is to say, the interior nodes x_k are the zeros of $\frac{d}{dx} J_{n+1}^{\alpha,\beta}(x)$.

Therefore, we can compute the eigenvalues of Jacobi matrix $\hat{J}_n$ to get the n interior nodes x_k.

4. Weights of the quadrature rule. Now, we restrict our attention to the weights of Gauss-Lobatto formulae. For the boundary weights, a lot of the results have been gotten, we refer the reader to Cautschi[4] and Yang[12], and they both derived the same formula for the boundary weights in a different way. Here, we give the result following Cautschi.

Since the change of variable $t \mapsto -t$ convert $d\lambda^{\alpha,\beta}$ into $d\lambda^{\beta,\alpha}$, one easily sees that $\lambda_{n+1}^{\alpha,\beta} = \lambda_0^{\beta,\alpha}$, It suffices therefore to compute $\lambda_0^{\alpha,\beta}$ and it can be expressible explicitly as

$$\lambda_0^{\alpha,\beta} = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+3)} \frac{\binom{n+\alpha+1}{n}}{\binom{n+\beta+1}{n}\binom{n+\alpha+\beta+2}{n}}$$

and $\lambda_{n+1}^{\alpha,\beta} = \lambda_0^{\beta,\alpha}$.

Gautschi and Yang give their formula for the interior weights respectively, and they are different in the form. Cautschi's formula seems to be quite complicated, while Yang's is simpler comparing with the former, but not practical in the actual computation. Below are their formulas for the interior weights.
Gautschi's :

$$\lambda_k^{\alpha,\beta} = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+2)\Gamma(\beta+2)}{\Gamma(\alpha+\beta+3)(n+1)^2} \frac{\binom{n+\alpha+1}{n}\binom{n+\beta+1}{n}}{\binom{n+\alpha+\beta+2}{n}} \times \frac{1}{\frac{4(n+\alpha+1)(n+\beta+1)+(\alpha-\beta)^2}{(2n+\alpha+\beta+2)^2} - x_k^2} \frac{1-x_k^2}{\left[J_{n+1}^{\alpha,\beta}(x_k)\right]^2}$$

Yang's:

$$\lambda_k^{\alpha,\beta} = 2^{\alpha+\beta+3} \frac{\Gamma(\alpha+n+2)\Gamma(\beta+n+2)}{\Gamma(n+1)\Gamma(\alpha+\beta+n+3)\left[(1-x_k^2)J_n^{(\alpha+1,\beta+1)'}(x_k)\right]^2}$$

where $J_n^{(\alpha+1,\beta+1)'}(x_k)$ denote as the value of derivative of $J_n^{\alpha+1,\beta+1}(x)$ at x_k.

The second aim of this paper is seek more compact form and more practical for computation.

Theorem 4.1 The interior weights associated with the interior nodes listed as follows:

$$\lambda_k^{\alpha,\beta}=2^{\alpha+\beta+1}\frac{\Gamma(\alpha+n+2)\Gamma(\beta+n+2)}{(n+1)^2\Gamma(n+1)\Gamma(\alpha+\beta+n+3)\left[J_{n+1}^{\alpha,\beta}(x_k)\right]^2} \tag{4.1}$$

Proof: Thanks to (3.6), We rewrite it as (replacing n by n+1)

$$\frac{d}{dx}J_{n+1}^{\alpha,\beta}(x)=\frac{1}{2}(n+\alpha+\beta+2)J_n^{\alpha+1,\beta+1}(x). \tag{4.2}$$

As mentioned above, the interior nodes x_k are the zeros of $J_n^{\alpha+1,\beta+1}(x)$, namely

$$\frac{d}{dx}J_{n+1}^{\alpha,\beta}(x_k)=J_n^{\alpha+1,\beta+1}(x_k)=0, \qquad k=1,2,\cdots,n. \tag{4.3}$$

Differentiate (4.2) to get

$$\frac{d^2}{dx^2}J_{n+1}^{\alpha,\beta}(x)=\frac{1}{2}(n+\alpha+\beta+2)\frac{d}{dx}J_n^{\alpha+1,\beta+1}(x) \tag{4.4}$$

Recall that Jacobi polynomials $J_{n+1}^{\alpha,\beta}(x)$ satisfy Sturm-Liouville Eq.(3.3) (replacing n as n+1)

$$(1-x^2)\frac{d^2}{dx^2}J_{n+1}^{\alpha,\beta}(x)+\big(\beta-\alpha-(\beta+\alpha+2)x\big)\frac{d}{dx}J_{n+1}^{\alpha,\beta}(x)+(n+1)(n+2+\alpha+\beta)J_{n+1}^{\alpha,\beta}(x)=0. \tag{4.5}$$

Substitute (4.4) into (4.5), we can get

$$\frac{1}{2}(n+\alpha+\beta+1)(1-x^2)\frac{d}{dx}J_n^{\alpha+1,\beta+1}(x)+\big(\beta-\alpha-(\beta+\alpha+2)x\big)\frac{d}{dx}J_{n+1}^{\alpha,\beta}(x)+n(n+1+\alpha+\beta)J_{n+1}^{\alpha,\beta}(x)=0.$$

Using the fact (4.3), the above formulae can be simplified as

$$(1-x_k^2)\frac{d}{dx}J_n^{\alpha+1,\beta+1}(x_k)=-2(n+1)J_{n+1}^{\alpha,\beta}(x_k) \tag{4.6}$$

With Yang's formula and Eq. (5.6), we develop a more simple and practical formula for the interior weights in the actual computation

$$\lambda_k^{\alpha,\beta}=2^{\alpha+\beta+1}\frac{\Gamma(\alpha+n+2)\Gamma(\beta+n+2)}{(n+1)^2\Gamma(n+1)\Gamma(\alpha+\beta+n+3)\left[J_{n+1}^{\alpha,\beta}(x_k)\right]^2}.$$

This completes the proof.

This is true for Legendre case with $\alpha=\beta=0$. In fact,

$$\lambda_k^{0,0}=2\frac{\Gamma(n+2)\Gamma(n+2)}{(n+1)^2\Gamma(n+1)\Gamma(n+3)\left[J_{n+1}^{0,0}(x_k)\right]^2}=\frac{2}{(n+1)(n+2)\left[L_{n+1}(x_k)\right]^2}$$

5. Numerical Experiments and Analysis. In the numerical experiment described in this section, we compare our methods based on newly-established recursive relation for the derivatives of Jacobi polynomials with conventional modified Jacobi matrix for Gauss-Lobtto nodes, and the associated weights are also considered.

We begin with the classical Chebyshev-Gauss-Lobatto formula, i.e. with the case $\alpha = \beta = -\frac{1}{2}$, whose nodes and weights are known explicitly

$$x_k = \cos(\frac{k\pi}{n+1}), \lambda_k = \frac{\pi}{\tilde{c}_k (n+1)}, k = 0,1,\cdots,n+1$$

where $\tilde{c}_0 = \tilde{c}_{n+2} = 2,\ and \quad \tilde{c}_k = 1, k = 1,2,\cdots,n$

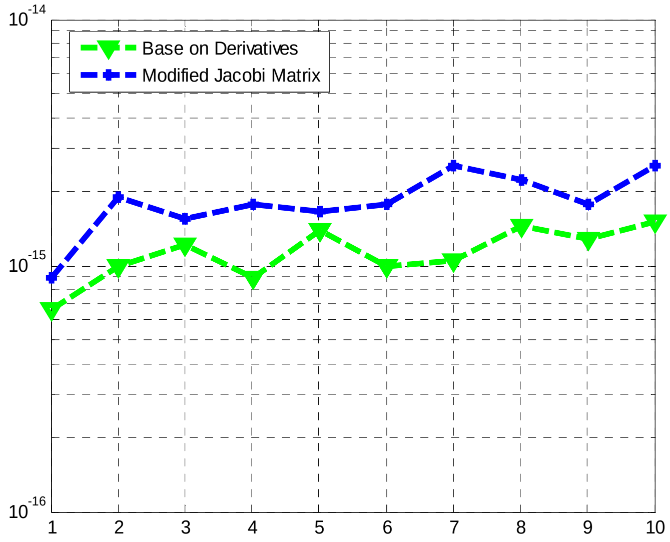

Fig. 5.1 Nodes comparison of method based derivative and modified Jacobi matrix for n=100 to 1000 per 100 in the case α=β=–0.5

The absolute errors of our method and modified Jacobi matrix with explicit formula[4] for nodes of Chebyshev-Gauss-Lobatto compared with the exact formula are plotted in the Fig.5.1 ,where the green line denote the our method, and the blue line Gautschi.

Fig.5.1 demonstrates that when the newly-established recursive relation for derivative of Jacobi polynomials is used for computing the Gauss-Lobatto nodes by QR algorithm, the results will be more robust and efficient compare to the method of based-on modified Jacobi matrix improved by Gautschi. We will be more intended to use robust method in the computation to some degree.

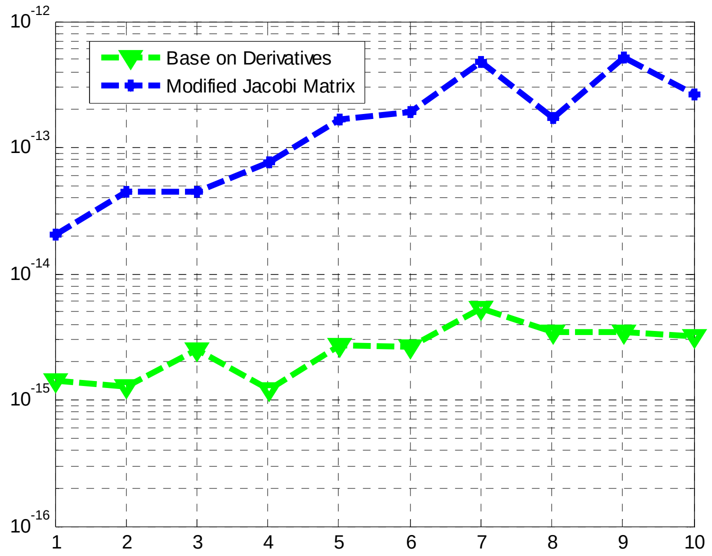

Fig.5.2 Weights comparison of method based derivative and modified Jacobi matrix for n=100 to 1000 per 100 in the case α=β=–0.5

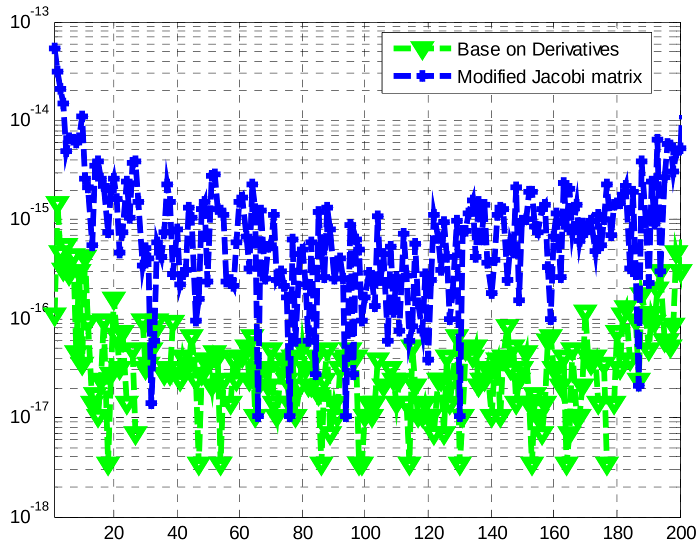

Fig.5.3 Error of weights per component based derivative and modified Jacobi matrix when n=200 in the case α=β=–0.5

Comparison of weights is also plotted in the Fig.5.2. The color of line stands as before. From it, we can see results based on our explicit formula (4.1) are more accuracy than the results given by Gautschi's explicit formula. The formula shows us a more stable property for computing weights. Fig.5.3 is used as the reinterpretation of the interior weights with great robust and accuracy in the actual computation.

This paper mainly derive the three-term recurrence relation for the derivatives of Jacobi polynomials which is new to our best knowledge, and compute eigenvalues of the special Jacobi matrix of order n which is different from the modified Jacobi matrix by Golub and Gautschi. Then we develop another simpler explicit expression and practical for the computation. Numerical experiments verify that our newly-established recursive relation and explicit weight formula is efficient and robust superior to other methods.

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