Using algebraic multigrid for lithospheric-scale numerical modelling

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Summary

Various types of multigrid methods have been successfully applied in the mantle convection simulation in recent decades (see e.g. Tackley, 2008). These methods traditionally belong to the geometric multigrid category, with all its known limitations with respect to the localized material discontinuities, and necessity to explicitly define a sequence of coarse grids. As an attempt to circumvent these disadvantages, a few researches have recently turned to another well-developed category of methods: the so-called algebraic multigrid (see e.g. Burstedde et al., 2008; Geenen et al., 2009). Up to now the application of the algebraic multigrid methods has been limited to the mantle convection simulation, and remains mostly a conceptual research area. It is well-known that in the lithospheric domain, the requirements to handle the localized material discontinuities are even more pronounced than in the mantle. Therefore, in this work I will try to extend current experience, and demonstrate the results of application of the algebraic multigrid to a conceptual 3D continental breakup problem.

The algebraic multigrid is not usually used as a stand-alone linear solver, but mostly as a preconditioner for the Krylov-type iteration, applied either to a coupled velocity-pressure system, or just to the pressure Schur complement. The latter approach requires highly accurate solutions of the inner velocity subsystems (e.g. May and Moresi, 2008), which generally has negative impact on the computational time. Because of this shortcoming, I will only use the coupled velocity-pressure solution strategy in this work. Recent unpublished studies of myself, and personal communications with other researches have confirmed that using any type of the finite elements with continuous pressure, stabilization, or even some types of the LBB-stable elements (e.g. Fortin, 1981), produce erroneous results in computational geodynamics applications. For the sake of time saving and reliability of the models, the calculations are done using $Q_2 – P_{-1}$ hexahedral elements. The tetrahedral $P^+_2 – P_{-1}$ (Crouzeix-Raviart) element remains a valuable alternative, in case that fully unstructured grid is desired.

I use essentially the same computational framework as published in Popov and Sobolev (2008). The modified code solves coupled mass, momentum and energy conservation equations in incremental displacements, incremental pressure and temperature primitive variables. Improvements include implementation of the $Q_2 – P_{-1}$ element, separation of pressure as independent variable, and using Trilinos package (e.g. Heroux et al., 2005) for parallel scalable implementation of algebraic multigrid ingredients. Namely, I use Epetra package for distributed sparse matrices and vectors, AztecOO for parallel Krylov solvers, and ML package for the smoothed aggregation multigrid preconditioning. All these packages use the same data structure, and can be easily coupled together in a single object-oriented application.

The crucial point of the coupled iterative velocity-pressure solver is, of course, the effective preconditioner. For review and discussion of some possible approaches in geodynamics see e.g. May and Moresi (2008). Usually the preconditioner is constructed based on a block factorization of the linear system, with subsequent dropping/approximation of certain diagonal and off-diagonal blocks. Using the popular block-triangular preconditioner necessitates the use of a non-symmetric Krylov solver such as GMRES or GCR, which imply an overhead of additional storage for Krylov basis. I try to keep symmetry in the preconditioner instead, but not at the cost of dropping the off-

diagonal blocks. Because the pressure submatrix has small but generally nonzero values due to compressibility, I can apply the less costly traditional Conjugate Gradient solver instead of MINRES. The inversion of either the block-diagonal approximation, or the full velocity (displacements) submatrix in the preconditioner is done using fixed number of algebraic multigrid V-cycles, or by inexact CG solver with multigrid as preconditioner. The latter approach requires the use of a flexible variant of the CG algorithm for the coupled iteration. The pressure Schur complement preconditioner ranges from a simple pressure mass matrix scaled by inverse viscosity, to the advanced BFBt preconditioner (e.g. Elman, 2005). I also explicitly construct algebraic approximations for the Schur complement, and invert them using algebraic multigrid. My presentation will include the comparison of the above mentioned techniques and their applicability to the representative lithospheric-scale 3D model.

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